

On minimal spheres of area 4π and rigidity

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Abstract

Let M be a complete Riemannian 3-manifold with sectional curvatures between 0 and 1. A minimal 2-sphere immersed in M has area at least 4π . If an embedded minimal sphere has area 4π , then M is isometric to the unit 3-sphere or to a quotient of the product of the unit 2-sphere with \mathbb{R} , with the product metric. We also obtain a rigidity theorem for the existence of hyperbolic cusps. Let M be a complete Riemannian 3-manifold with sectional curvatures bounded above by -1 . Suppose there is a 2-torus T embedded in M with mean curvature one. Then the mean convex component of M bounded by T is a hyperbolic cusp; *i.e.*, it is isometric to $T \times \mathbb{R}$ with the constant curvature -1 metric: $e^{-2t}d\sigma_0^2 + dt^2$ with $d\sigma_0^2$ a flat metric on T .

Keywords: area of minimal sphere, rigidity of 3-manifolds, hyperbolic cusp.

1 Introduction

Consider a smooth (C^∞) complete metric on the 2-sphere S whose curvature is between 0 and 1. It is well known that a simple closed geodesic in S has length at least 2π (see [4] or Klingenberg's theorem in higher dimension [3, 2]). It is less well known that when such an S has a simple closed geodesic of length exactly 2π , then S is isometric to the unit 2-sphere \mathbb{S}_1^2 . This result is proved in [1], and the authors attribute the theorem to E. Calabi.

With this in mind, we consider what happens in a complete 3-manifold M with sectional curvatures between 0 and 1 (henceforth we suppose this curvature condition on M , unless stated otherwise).

Let Σ be an embedded minimal 2-sphere in M . Then the Gauss-Bonnet theorem and the Gauss equation tells us that the area of S is at least 4π : indeed we have

$$4\pi = \int_{\Sigma} \bar{K}_{\Sigma} = \int_{\Sigma} \det(A) + K_{T\Sigma} \leq \int_{\Sigma} 1 = A(\Sigma) \quad (1)$$

with $\det(A)$ the determinant of the shape operator which is non positive. We prove in Theorem 1, that when the area of Σ equals 4π , then M is isometric to the unit 3-sphere \mathbb{S}_1^3 or to a quotient of the product of the unit 2-sphere with \mathbb{R} , $\mathbb{S}_1^2 \times \mathbb{R}$, with the product metric.

We remark that Theorem 1 does not hold for embedded minimal tori. Given ε greater than zero, there are Berger spheres with curvatures between 0 and 1, which contain embedded minimal tori of area less than ε . But a minimal sphere always has area at least 4π .

It would be interesting to know what happens in higher dimensions. In the unit n -sphere \mathbb{S}_1^n , a compact minimal hyper-surface Σ always has volume at least the volume of the equatorial $n - 1$ sphere \mathbb{S}_1^{n-1} . Is there a rigidity theorem when one allows metrics on \mathbb{S}^n ($= M$), of sectional curvatures between 0 and 1? Two questions arise. First, does an embedded minimal hyper-sphere Σ in M have volume at least the volume of \mathbb{S}_1^{n-1} . If this is so, and if Σ is an embedded minimal hyper-sphere with volume exactly the volume of \mathbb{S}_1^{n-1} , is M isometric to \mathbb{S}_1^n or to $\mathbb{S}_1^{n-1} \times \mathbb{R}$?

In the same spirit as Theorem 1, we prove a rigidity theorem for hyperbolic cusps. We recall that a 3 dimensional hyperbolic cusp is a manifold of the form $T \times \mathbb{R}$ with T a 2-torus and the hyperbolic metric $e^{-2t}d\sigma_0^2 + dt^2$ with $d\sigma_0^2$ a flat metric on T . In Theorem 2, we prove that if M is a complete Riemannian manifold with sectional curvatures bounded above by -1 and T is a constant mean curvature 1 torus embedded in M then the mean convex side of T in M is isometric to a hyperbolic cusp.

2 Minimal spheres of area 4π and rigidity of 3-manifolds

In this section, we prove a rigidity result for a Riemannian 3-manifold M whose sectional curvatures are between 0 and 1. As explained in the introduction, any minimal sphere in such a manifold has area at least 4π .

We denote by \mathbb{S}_1^n the sphere of dimension n with constant sectional curvature 1. We then have the following result.

Theorem 1. *Let M be a complete Riemannian 3-manifold whose sectional curvatures satisfy $0 \leq K \leq 1$. Assume that there exists an embedded minimal sphere Σ in M with area 4π . Then the manifold M is isometric either to the sphere \mathbb{S}_1^3 or to a quotient of $\mathbb{S}_1^2 \times \mathbb{R}$.*

Proof. Let Φ be the map $\Sigma \times \mathbb{R} \rightarrow M, (p, t) \mapsto \exp_p(tN(q))$ where N is a unit normal vector field along Σ . In the following, we focus on $\Sigma \times \mathbb{R}_+$; by

symmetry of the configuration, the study is similar for $\Sigma \times \mathbb{R}_-$.

Σ is compact, so there is an ε such that Φ is an immersion and even an embedding on $\Sigma \times [0, \varepsilon)$. Let us define

$$\varepsilon_0 = \sup\{\varepsilon > 0 \mid \Phi \text{ is an immersion on } \Sigma \times [0, \varepsilon)\};$$

ε_0 can be equal to $+\infty$. Using Φ , we pull back the Riemannian metric of M to $\Sigma \times [0, \varepsilon_0)$. This metric can be written $ds^2 = d\sigma_t^2 + dt^2$ where $d\sigma_t^2$ is a smooth family of metrics on Σ . With this metric, Φ becomes a local isometry from $\Sigma \times [0, \varepsilon_0)$ to M and $(\Sigma \times [0, \varepsilon_0), ds^2)$ has sectional curvatures between 0 and 1. Moreover, Σ_0 is minimal and has area 4π . Actually, we will prove the following facts.

Claim. *The metric $d\sigma_0^2$ has constant sectional curvature 1 so $(\Sigma, d\sigma_0^2)$ is isometric to \mathbb{S}_1^2 . Moreover, we have two cases*

1. $\varepsilon_0 = \pi/2$ and $d\sigma_t^2 = \sin^2 t d\sigma_0^2$ or
2. $\varepsilon_0 = +\infty$ and $d\sigma_t^2 = d\sigma_0^2$

Let us denote by $\Sigma_t = \Sigma \times \{t\}$ the equidistant surfaces. We denote by $H(p, t)$ the mean curvature of Σ_t at the point (p, t) with respect to the unit normal vector ∂_t . We also define $\lambda(p, t) \geq 0$ such that $H + \lambda$ and $H - \lambda$ are the principal curvature of Σ_t at (p, t) . We notice that $\lambda = 0$ if Σ_t is umbilical at (p, t) .

The surfaces Σ_t are spheres so, using the Gauss equation, the Gauss-Bonnet formula implies:

$$4\pi = \int_{\Sigma_t} \bar{K}_{\Sigma_t} = \int_{\Sigma_t} (H + \lambda)(H - \lambda) + K_t = \int_{\Sigma_t} H^2 - \lambda^2 + K_t$$

where \bar{K}_{Σ_t} is the intrinsic curvature of Σ_t and K_t is the sectional curvature of the ambient manifold of the tangent space to Σ_t . Since $K_t \leq 1$, we obtain the following inequality

$$\int_{\Sigma_t} \lambda^2 = \int_{\Sigma_t} H^2 + K_t - 4\pi \leq \int_{\Sigma_t} H^2 + A(\Sigma_t) - 4\pi \quad (2)$$

where $A(\Sigma_t)$ is the area of Σ_t . In the following, we denote by $F(t)$ the right hand side of this inequality.

Claim 1. *F is vanishing on $[0, \varepsilon_0)$.*

Since Σ_0 is minimal and has area 4π , we have $F(0) = 0$. We notice that this implies that $\lambda(p, 0) = 0$ so Σ_0 is umbilical and $K_{T\Sigma_0} = 1$. Thus $(\Sigma_0, d\sigma_0)$ is isometric to \mathbb{S}_1^2 .

We have the usual formula:

$$\frac{\partial}{\partial t} A(\Sigma_t) = - \int_{\Sigma_t} 2H \quad \text{and} \quad \frac{\partial H}{\partial t} = \frac{1}{2}(\text{Ric}(\partial_t) + |A_t|^2) \quad (3)$$

where A_t is the shape operator of Σ_t and Ric is the Ricci tensor of $\Sigma \times [0, \varepsilon_0)$. Since the sectional curvatures of $M \times [0, \varepsilon_0)$ are non-negative, Ric is non-negative. So the second formula above implies that H is increasing and thus $H \geq 0$ everywhere. Let us now compute and estimate the derivative of F :

$$\begin{aligned} F'(t) &= \int_{\Sigma_t} (2H \frac{\partial H}{\partial t} - 2H^3) - \int_{\Sigma_t} 2H \\ &= \int_{\Sigma_t} H(\text{Ric}(\partial_t) + |A_t|^2 - 2H^2 - 2) \\ &= \int_{\Sigma_t} H((\text{Ric}(\partial_t) - 2) + ((H + \lambda)^2 + (H - \lambda)^2 - 2H^2)) \\ &= \int_{\Sigma_t} H((\text{Ric}(\partial_t) - 2) + 2\lambda^2) \\ &\leq 2 \int_{\Sigma_t} H\lambda^2 \end{aligned}$$

where the last inequality comes from $\text{Ric}(\partial_t) - 2 \leq 0$ because of the hypothesis on the sectional curvatures. If we choose $\varepsilon < \varepsilon_0$, there is a constant $C \geq 0$ such that $H \leq C$ on $\Sigma \times [0, \varepsilon]$. So for $t \in [0, \varepsilon]$, using the inequality (2), we get $F'(t) \leq 2CF(t)$. Then $F(t) \leq F(0)e^{2Ct} = 0$ on $[0, \varepsilon]$. So $F \leq 0$ on $[0, \varepsilon_0)$ and, because of (2), $F = 0$ on $[0, \varepsilon_0)$; this finishes the proof of Claim 1.

The first consequence of Claim 1 is that all the equidistant surfaces Σ_t are umbilical (see inequality (2)); so $\lambda \equiv 0$. In the computation of the derivative of F , this implies that

$$\int_{\Sigma_t} H(\text{Ric}(\partial_t) - 2) = 0$$

Since $H(\text{Ric}(\partial_t) - 2) \leq 0$ everywhere, we obtain

$$H(\text{Ric}(\partial_t) - 2) = 0 \text{ everywhere.} \quad (4)$$

Moreover the umbilicity and (3) implies that $\frac{\partial H}{\partial t} = \frac{1}{2}\text{Ric}(\partial_t) + H^2$. We now prove the following claim

Claim 2. *Let $(p, t) \in \Sigma \times [0, \varepsilon_0)$ ($t > 0$) be such that $H(p, t) > 0$ then $H(q, t) > 0$ for any $q \in \Sigma$*

In other words, when the mean curvature is positive at a point of an equidistant, it is positive at any point of this equidistant. We recall that H is increasing in the t variable so when it becomes positive it stays positive.

So assume that $H(p, t) > 0$ and consider $\Omega = \{q \in \Sigma \mid H(q, t) > 0\}$ which is a nonempty open subset of Σ . Let $q \in \Omega$. Since $H(q, t) > 0$, $Ric(\partial_t)(q, t) = 2$ by (4). Thus $Ric(\partial_t)(r, t) = 2$ for any $r \in \bar{\Omega}$. So if $r \in \bar{\Omega}$, $Ric(\partial_t)(r, s) > 0$ for $s < t$, close to t and, by (3), this implies that $H(r, t) > 0$ and $r \in \Omega$. So Ω is closed and $\Omega = \Sigma$. This finishes the proof of Claim 2.

Let us assume that there is an $\varepsilon_1 > 0$ such that $H(p, t) = 0$ for $(p, t) \in \Sigma \times [0, \varepsilon_1]$ and $H(p, t) > 0$ for any $(p, t) \in \Sigma \times (\varepsilon_1, \varepsilon_0)$. Because of the evolution equation of H , this implies that $Ric(\partial_t) = 0$ on $\Sigma \times [0, \varepsilon_1]$. On $\Sigma \times (\varepsilon_1, \varepsilon_0)$, we have $Ric(\partial_t) = 2$ because of (4). So by continuity of $Ric(\partial_t)$, we get a contradiction and then we have two possibilities

1. $H = 0$ on $\Sigma \times [0, \varepsilon_0)$ and $Ric(\partial_t) = 0$ on $\Sigma \times [0, \varepsilon_0)$.
2. $H > 0$ on $\Sigma \times (0, \varepsilon_0)$ and $Ric(\partial_t) = 2$ on $\Sigma \times [0, \varepsilon_0)$.

In the first case, this implies that the sectional curvature of any 2-plane orthogonal to Σ_t is zero. Thus $d\sigma_t^2 = d\sigma_0^2$. Since the map Φ ceases to be an immersion only if $d\sigma_t^2$ becomes singular this implies that $\varepsilon_0 = +\infty$. Thus $\Sigma \times \mathbb{R}_+$ with the induced metric is isometric to $\mathbb{S}_1^2 \times \mathbb{R}_+$ and Φ is a local isometry from $\mathbb{S}_1^2 \times \mathbb{R}_+$ to M .

In the second case, the sectional curvature of any 2-plane orthogonal to Σ_t is equal to 1. Thus $d\sigma_t^2 = \sin^2 t d\sigma_0$ and $\varepsilon_0 = \pi/2$. This also implies that $\Phi(p, \pi/2)$ is a point. So $\Sigma \times [0, \pi/2]$ with the metric ds^2 is isometric to a hemisphere of \mathbb{S}_1^3 and the map Φ is a local isometry from that hemisphere to M .

Doing the same study for $\Sigma \times \mathbb{R}_-$, we get in the first case a local isometry $\Phi : \mathbb{S}_1^2 \times \mathbb{R} \rightarrow M$ and in the second case a local isometry $\Phi : \mathbb{S}_1^3 \rightarrow M$. Since $\mathbb{S}_1^2 \times \mathbb{R}$ and \mathbb{S}_1^3 are simply connected, Φ is then the universal cover of M and M is then isometric to a quotient of $\mathbb{S}_1^2 \times \mathbb{R}$ or \mathbb{S}_1^3 . Since Φ is injective on Σ this implies that in the second case, Φ is actually injective and then a global isometry. \square

Remark 1. In the proof, since Φ is injective on Σ , the possible quotients of $\mathbb{S}_1^2 \times \mathbb{R}$ are either $\mathbb{S}_1^2 \times \mathbb{R}$ or its quotient by the subgroup generated by an isometry of the form $\mathbb{S}_1^2 \times \mathbb{R} \rightarrow \mathbb{S}_1^2 \times \mathbb{R}; (p, t) \mapsto (\alpha(p), t + t_0)$ with α an isometry of \mathbb{S}_1^2 and $t_0 \neq 0$.

Remark 2. Something can be said about constant mean curvature H_0 spheres in a Riemannian 3-manifold with sectional curvatures between 0 and 1. Indeed, the computation (1) implies that the area of Σ is larger than $\frac{4\pi}{1+H_0^2}$, which is the area of a geodesic sphere in \mathbb{S}_1^3 of mean curvature H_0 . Moreover, if Σ has area $\frac{4\pi}{1+H^2}$, the above proof can be adapted to prove that the mean convex side of Σ is isometric to a spherical cap of \mathbb{S}_1^3 with constant mean curvature H_0 (see Theorem 2 below, for a similar result in the hyperbolic case).

Remark 3. Let M be a Riemannian n -manifold whose sectional curvatures are between 0 and 1 and let Σ be a minimal 2-sphere in M . A computation similar to (1) proves also that the area of Σ is larger than 4π . It also implies that, if Σ has area 4π , Σ is totally geodesic and isometric to \mathbb{S}_1^2 .

3 Existence of hyperbolic cusps

Let (\mathbb{T}^2, g) be a flat 2 torus, the manifold $\mathbb{T}^2 \times \mathbb{R}_+$ with the complete Riemannian metric $e^{-2t}g + dt^2$ is a hyperbolic 3-dimensional cusp. $\mathbb{T}^2 \times \mathbb{R}$ is actually isometric to the quotient of a horoball of \mathbb{H}^3 by a \mathbb{Z}^2 subgroup of isometries of \mathbb{H}^2 leaving the horoball invariant. Any $\mathbb{T}^2 \times \{t\}$ has constant mean curvature 1. The following theorem says that, in certain 3-manifolds, a constant mean curvature 1 torus is necessarily the boundary of a hyperbolic cusp.

Theorem 2. *Let M be a complete Riemannian 3-manifold with its sectional curvatures satisfying $K \leq -1$. Assume that there exists a constant mean curvature 1 torus T embedded in M . Then T separates M and its mean convex side is isometric to a hyperbolic cusp.*

As a consequence, the existence of this torus implies that M can not be compact. The proof uses the same ideas as in Theorem 1

Proof. Let us consider the map $\Phi : T \times \mathbb{R}_+ \rightarrow M, (p, t) \mapsto \exp_p(tN(p))$ where N is the unit normal vector field normal to T such that N is the mean curvature vector of T . Let us define

$$\varepsilon_0 = \sup\{\varepsilon > 0 \mid \Phi \text{ is an immersion on } T \times [0, \varepsilon)\}.$$

Using Φ , we pull back the Riemannian metric of M to $T \times [0, \varepsilon_0)$; it can be written $ds^2 = dt^2 + d\sigma_t^2$. We define $T_t = T \times \{t\}$ the equidistant surfaces to T_0 . We also denote by $H(p, t)$ the mean curvature of the equidistant surfaces

at (p, t) with respect to ∂_t . We finally define $\lambda(p, t)$ such that $H + \lambda$ and $H - \lambda$ are the principal curvatures of T_t at (p, t) .

The surfaces T_t are tori so, by the Gauss equation and the Gauss-Bonnet formula, we have

$$0 = \int_{T_t} \bar{K}_{T_t} = \int_{T_t} H^2 - \lambda^2 + K_t$$

where K_t is the sectional curvature of the ambient manifold of the tangent space to T_t . Since $K_t \leq -1$, we obtain the inequality

$$\int_{T_t} \lambda^2 = \int_{T_t} H^2 + K_t \leq \int_{T_t} H^2 - A(T_t)$$

Let $F(t)$ denote the right hand term of the above inequality. By hypothesis, $H(p, 0) = 1$ so $F(0) = 0$ and $F(t) \geq 0$ for any $t \geq 0$. Let us compute the derivative of F

$$\begin{aligned} F'(t) &= \int_{T_t} (2H \frac{\partial H}{\partial t} - 2H^3) + \int_{T_t} 2H \\ &= \int_{T_t} H(Ric(\partial_t) + |A_t|^2 - 2H^2 + 2) \\ &= \int_{T_t} H((Ric(\partial_t) + 2) + 2\lambda^2) \end{aligned}$$

Since $H(p, 0) = 1$, we can consider $\varepsilon \in (0, \varepsilon_0)$ such that $0 < H \leq C$ on $T \times [0, \varepsilon]$. Since $Ric(\partial_t) + 2 \leq 0$ we get:

$$F'(t) \leq \int_{T_t} 2H\lambda^2 \leq 2CF(t)$$

Thus $F(t) \leq F(0)e^{2Ct}$ for $t \in [0, \varepsilon]$; this implies $F(t) = 0$ on that segment. We then obtain $\lambda = 0$ on $T \times [0, \varepsilon]$ (the equidistant surfaces are umbilical) and $Ric(\partial_t) = -2$ since $H > 0$. Thus H satisfies the differential equation $\frac{\partial H}{\partial t} = -2 + 2H^2$. This gives that $H = 1$ on $T \times [0, \varepsilon]$ since $H = 1$ on T_0 . Thus we can let ε tend to ε_0 to obtain that $F(t) = 0$ on $[0, \varepsilon_0)$ and $Ric(\partial_t) = -2$ and $H = 1$ on $T \times [0, \varepsilon_0)$. Since $0 = \int_{T_t} H^2 + K_t$ and $K_t \leq -1$, it follows that $K_t = -1$ for all t in the interval. We then have proved that the sectional curvature of $T \times [0, \varepsilon_0)$ with the metric ds^2 is equal to -1 for any 2-plane. Moreover, we get that $d\sigma_0^2$ is flat and that $d\sigma_t^2 = e^{-2t}d\sigma_0^2$. This implies that Φ is actually an immersion on $T \times \mathbb{R}_+$ ($\varepsilon_0 = +\infty$) and $T \times \mathbb{R}_+$ is isometric to a hyperbolic cusp. Φ is then a local isometry from this hyperbolic cusp to M .

To finish the proof, let us prove that Φ is in fact injective. If this is not the case, let $\varepsilon_1 > 0$ be the smallest ε such that Φ is not injective on $T \times [0, \varepsilon]$. This implies that there exist p and q in T such that

- either $\Phi(p, 0) = \Phi(q, \varepsilon_1)$
- or $\Phi(p, \varepsilon_1) = \Phi(q, \varepsilon_1)$ (with $p \neq q$ in this case).

Let U and V be respective neighborhoods of $(p, 0)$ (or (p, ε_1)) in T_0 (or T_{ε_1}) and (q, ε_1) in T_{ε_1} such that Φ is injective on them. Since ε_1 is the smallest one, $\Phi(U)$ and $\Phi(V)$ are two constant mean curvature 1 surfaces in M that are tangent at $\Phi(q, \varepsilon_1)$. Moreover, in the first case, $\Phi(U)$ is included in the mean convex side of $\Phi(V)$ so by the maximum principle $\Phi(U) = \Phi(V)$. Thus $\Phi(T_0)$ would be equal to $\Phi(T_{\varepsilon_1})$ which is impossible since these two surfaces do not have the same area. In the second case, $\Phi(U)$ is included in the mean convex side of $\Phi(V)$ and then Φ is not injective on T_s for s near t $s < t$, which is a contradiction. \square

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